1. Choice, Preferences, Utility

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Overview

1. Why Economic Theory
2. Choice and Preferences
3. Preferences and Utility
4. Limited Observability
5. More

Overview

Economic Theory

Goals

- Studies behaviour
- Understand how different forces interact and lead to different outcomes
- Positive view: Explain patterns, make predictions
- Normative view: Prescribe behaviour
- Examples: consumer demand and firm pricing, student applications to university, voting, technology adoption, hospital residency program management

(Not particular to theory: in essence, all science strives for generality)

This course

- Develop building blocks

Representing Behaviour

Choices, Preferences, Utility

Basic model: choices described by utility maximisation

agents choose an alternative x from a set of feasible alternatives S to maximize their utility u

Properties of u

u carries several implications for behaviour (warranted or not)

Undertanding implications often allows *testing* model through its *identifying* assumptions

Models as maps, simplified description of reality

Behavioural implications = Empirical content

Choice, Preferences, Utility

The elephant in the room

Economics "does study human beings, but only as entities having certain patterns of market behaviour, it makes no claim, no pretence, to be able to see inside their heads" (Hicks 1956)

Behaviour is driven by taste, pleasure, and gratification,
by notions of duty and consideration for others,
by reason, strategy, deduction,
by distraction, habit, biological determinants, emotion, impulse

This course: model behaviour

Terminology is *technical*: 'preference', 'utility', 'rational', 'better', etc. have specific meanings

Preference and utility as mathematical objects used to represent behaviour (Samuelson 1938)

Utility/Preference **do not** have a welfaristic interpretation
(actions don't always increase well-being, but still worthwhile studying)

Overview

Choice

Finite set of alternatives X

$2^X := \{A \mid A \subseteq X\}$, all possible subsets of X

Model: choice from X

Definition

A **choice function** is a function $C : 2^X \to 2^X$ such that $C(A) \subseteq A \;\; \forall A \in 2^X$. We further require choice functions to be **nonempty**, that is, $\forall A \neq \emptyset, C(A) \neq \emptyset$.

Choice function determines agent's choices in every possible situation

Preferences

Preference relation on X

Binary relation $\succsim$ on X

- $\succsim \subseteq X \times X$
- $x \succsim y$ (or $y \precsim x$) equiv. to $(x, y) \in \succsim$

Definition

We say that a binary relation $\succsim$ on X is

- **reflexive** iff $\forall x \in X, x \succsim x$;
- **transitive** iff $\forall x, y, z \in X, x \succsim y$ and $y \succsim z$ implies $x \succsim z$;
- **negatively transitive** iff $\forall x, y, z \in X, x \succsim y$, then $x \succsim z$ or $z \succsim y$;
- **complete**[a] iff $\forall x, y \in X, x \succsim y$ or $y \succsim x$;
- **antisymmetric** iff $\forall x, y \in X, x \succsim y$ and $y \succsim x$ implies $x = y$;
- **symmetric** iff $\forall x, y \in X, x \succsim y$ implies $y \succsim x$;
- **asymmetric** iff $\forall x, y \in X, x \succsim y$ implies $\neg(y \succsim x)$.

[a] In order theory, especially outside economics, you may also find this property being called (strongly) connected, total, or connex.

Preferences

Definition

A binary relation $\succsim$ is called

(i) a **preorder** iff it is reflexive and transitive;

(ii) a **partial order** iff it is reflexive, transitive, and antisymmetric (an antisymmetric preorder);

(iii) a **linear order** (or total order) iff it reflexive, transitive, antisymmetric, and complete (a complete partial order).

$(X, \succsim)$: (i) preordered set; (ii) partially ordered set; (iii) linearly/totally ordered set

Examples

- (i) but not (ii)? population in different territories, ticket prices for different seats in a theatre, laptops ordered by price and specs (why?)
- (ii) but not (iii)? colours by RGB, *categories* of laptops ordered by price and specs, natural product order on $\mathbb{R}^n$
- (iii)? rank of items on a list, price *categories*, numeric ID numbers, natural order on $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

Preferences

Preference relation on X: complete and transitive

Terminology:

- *Weak preference*: $x \succsim y$
- *Indifference*: $x \sim y := x \succsim y$ and $y \succsim x$
  NB: $\sim \subseteq \succsim$ is the symmetric part of $\succsim$
  $x \sim y \not\Longrightarrow x = y$ (don't require antisymmetry)
- *Strict preference*: $x \succ y := x \succsim y$ and $\neg(y \succsim x)$
  NB: $\succ \subseteq \succsim$ is the asymmetric part of $\succsim$
- $\succsim = \succ \cup \sim$
  (can always decompose for any binary relation in sym. and asym. parts)

Proposition

A binary relation $\succsim \subseteq X \times X$ is complete and transitive only if its asymmetric part, $\succ \subseteq X \times X$, is asymmetric and negatively transitive.
A binary relation $\succ \subseteq X \times X$ is asymmetric and negatively transitive only if there is $\succsim \subseteq X \times X$ such that $\succ \subseteq \succsim$, $\succ$ is the asymmetric part of $\succsim$, and $\succsim$ is complete and transitive.

(Exercise in lecture notes)

Properties of $\arg\max_{\succsim} A$

For pref. rel. $\succsim \subseteq X^2$, define, for every $A \in \mathbf{2}^X$, set of $\succsim$-maximisers in A

$$\arg\max_{\succsim} A := \{x \in A \mid x \succsim y \text{ for all } y \in A\}$$

Proposition

Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{\succsim} A$ and $y \in \arg\max_{\succsim} B$, $x \succsim y$.

(ii) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{\succsim} A$, then $x \in \arg\max_{\succsim} B$.

(iii) For any nonempty $A \subseteq X$, $\arg\max_{\succsim} A \neq \emptyset$.

(iv) For $x, y \in A \subseteq X$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{\succsim} A$.

Properties of $\arg\max_{\succsim} A$

Proposition

Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{\succsim} A$ and $y \in \arg\max_{\succsim} B$, $x \succsim y$.

(ii) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{\succsim} A$, then $x \in \arg\max_{\succsim} B$.

Proof

(i) $x \in \arg\max_{\succsim} A \iff x \succsim z \, \forall z \in A$, and $y \in B \subseteq A$

(ii) As $x \in \arg\max_{\succsim} A \iff x \succsim z \ \forall z \in A$ and $B \subseteq A$,
then $x \succsim z \, \forall z \in B \iff x \in \arg\max_{\succsim} B$.

Properties of $\arg\max_{\succsim} A$

Proposition

Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(iii) For any nonempty $A \subseteq X$, $\arg\max_{\succsim} A \neq \emptyset$.

Proof

(iii) X is finite $\implies$ A is finite.

- $\forall A \in \mathbf{2}^X : |A| = \mathbf{1}$, then $A = \arg\max_{\succsim} A$ as $x \succsim x$ (by completeness) (hence $x \sim x$).
- Induction step: suppose $\forall B \in \mathbf{2}^X : B \neq \emptyset$ and $|B| = n \geq \mathbf{1}$, we have $\arg\max_{\succsim} B \neq \emptyset$. (true for $n = \mathbf{1}$)
- Take any $A \in \mathbf{2}^X : |A| = n + \mathbf{1}$; WTS $\arg\max_{\succsim} A \neq \emptyset$.
- $\exists B \in \mathbf{2}^A$ and $x \in X$ s.t. $A = B \cup \{x\}$, with $|B| = n$; also, for any $y, z \in \arg\max_{\succsim} B \neq \emptyset$, by completeness, $y \succsim x$ or $x \succsim y$.
- If $y \succsim x$, then $y \in \arg\max_{\succsim} A \because y \succsim z \; \forall z \in B$ and $y \succsim x$.
- If $x \succsim y$, then, as $y \in \arg\max_{\succsim} B \iff y \succsim z \; \forall z \in B$, transitivity implies $x \succsim z$ $\forall z \in B$, and hence $x \in \arg\max_{\succsim} A$.

Properties of $\arg\max_{\succsim} A$

Proposition

Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(iv) For $x, y \in A \subseteq X$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{\succsim} A$.

Proof

(iv) Let $\{x, y\} \subseteq A$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$.

- WLOG suppose $x \in \arg\max_{\succsim} A$.

$\Longrightarrow$: As $y \sim x \implies y \succsim x \succsim z \; \forall z \in A$, by transitivity $y \succsim z \; \forall z \in A \iff y \in \arg\max_{\succsim} A$.

$\Longleftarrow$: If $\{x, y\} \subseteq \arg\max_{\succsim} A$, then, by definition of $\arg\max_{\succsim}$, $x \succsim y$ and $y \succsim x$ ($\iff x \sim y$) and $x, y \in A$.

$\square$

Properties of $\arg\max_{\succsim} A$

Proposition

Let $\succsim \subseteq X \times X$ be a preference relation. The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{\succsim} A$ and $y \in \arg\max_{\succsim} B$, $x \succsim y$.

(ii) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{\succsim} A$, then $x \in \arg\max_{\succsim} B$.

(iii) For any nonempty $A \subseteq X$, $\arg\max_{\succsim} A \neq \emptyset$.

(iv) For $x, y \in A \subseteq X$, $x \sim y$ and $\{x, y\} \cap \arg\max_{\succsim} A \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{\succsim} A$.

Interpretation

(i): when set of feasible alternatives expands, preference relation attains weakly higher value.

(ii): if a $\succsim$-maximizer of a set A is also a $\succsim$-maximizer of any of its subsets. Often dubbed **independence of irrelevant alternatives** (IIA).

(iii): If set is finite, there is always $\succsim$-maximiser.
NB: if A not finite, then (iii) could fail (argmax could be empty); need more assumptions.

(iv): Indifference wrt any two maximisers.

Connecting Choice and Preferences: Revealed Preference

Definition (HARP)

A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies **Houthakker's Axiom of Revealed Preference** (HARP) if $\forall x, y \in X$, $\{x, y\} \subseteq A \cap B$, $x \in C(A)$ and $y \in C(B)$, then $x \in C(B)$ and $y \in C(A)$.

Oftentimes called *weak axiom of revelead preference*.

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Revealed preference: obtaining $\succsim$ from C (and vice-versa)

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be finite. Choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP $\iff \exists \succsim \subseteq X \times X : C(A) = \arg\max_{\succsim} A \ \forall A \in \mathbf{2}^X$.

Proof

$\implies$: (only if) Define $\succsim \subseteq X^2 : \forall x, y \in X, x \succsim y$ if $\exists A \in \mathbf{2}^X$ s.t. $x, y \in A$ and $x \in C(A)$.

- Completeness of $\succsim$:

  By definition of C, $\forall x, y \in X$, $C(\{x, y\}) \neq \emptyset$ and $C(\{x, y\}) \subseteq \{x, y\}$

  $\implies x \in C(\{x, y\}) \implies x \succsim y$ or $y \in C(\{x, y\}) \implies y \succsim x$.

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be finite. Choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP $\iff \exists \succsim \subseteq X \times X : C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Proof

$\implies$: (only if) Define $\succsim \subseteq X^2 : \forall x, y \in X, x \succsim y$ if $\exists A \in \mathbf{2}^X$ s.t. $x, y \in A$ and $x \in C(A)$.

- Transitivity:

  Let $x, y, z \in X$ s.t. $x \succsim y$ and $y \succsim z$; WTS $x \succsim z$.

  By definition of $\succsim$: $\exists A \ni x, y$ and $B \ni y, z$ s.t. $x \in C(A)$ and $y \in C(B)$.

  WTF $E \ni x, z$ and show $x \in C(E) \implies x \succsim z$ (by definition of $\succsim$). Take $E = \{x, y, z\}$.

(i) If $x \in C(\{x, y, z\})$, done.

(ii) If $y \in C(\{x, y, z\})$, as $x \in C(A)$ and $x, y \in A \cap \{x, y, z\}$,
HARP implies $x \in C(\{x, y, z\})$ and result follows.

(iii) If $z \in C(\{x, y, z\})$, as $y \in C(B)$ and $y, z \in B \cap \{x, y, z\}$,
HARP implies $y \in C(\{x, y, z\})$ and we are back to (ii).

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Proof

$\implies$: (only if) Define $\succsim \subseteq X^2 : \forall x, y \in X, x \succsim y$ if $\exists A \in \mathbf{2}^X$ s.t. $x, y \in A$ and $x \in C(A)$.

- WTS $C(A) = \arg\max_{\succsim} A, \forall A \in \mathbf{2}^X$.

$\subseteq$: WTS $C(A) \subseteq \arg\max_{\succsim} A$. Take $x \in C(A)$.

By definition of $\succsim$: $x \in C(A) \implies x \succsim y \; \forall y \in A$

By definition of $\arg\max_{\succsim} A$: $x \in \arg\max_{\succsim} A$; hence $C(A) \subseteq \arg\max_{\succsim} A$.

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Proof

$\implies$: (only if) Define $\succsim \subseteq X^2 : \forall x, y \in X, x \succsim y$ if $\exists A \in \mathbf{2}^X$ s.t. $x, y \in A$ and $x \in C(A)$.

- WTS $C(A) = \arg\max_{\succsim} A, \forall A \in \mathbf{2}^X$.

$\supseteq$: WTS $C(A) \supseteq \arg\max_{\succsim} A$. Take $x \in \arg\max_{\succsim} A \; (\subseteq A)$.

$\implies A \neq \emptyset$; hence $\exists y \in C(A)$ (choice functions on nonempty sets are nonempty).

Then $(x \in \arg\max_{\succsim} A$ and $y \in A) \implies x \succsim y$

$x \succsim y$ implies, by definition of $\succsim$, $\exists B \in \mathbf{2}^X$ s.t. $x, y \in B$ and $x \in C(B)$.

As $x, y \in A \cap B$, $x \in C(B)$ and $y \in C(A)$, by HARP, $x \in C(A)$

i.e.: $x \in \arg\max_{\succsim} A \implies x \in C(A)$.

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Proof

$\Longleftarrow$: (if) Define $C : \mathbf{2}^X \to \mathbf{2}^X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

- WTS: C is a choice function on X.

(i) WTS $C(A) \subseteq A$.

Follows by definition of $\arg\max_{\succsim}$

(ii) WTS $C(A) \neq \emptyset \; \forall A \neq \emptyset$.

Follows from property (ii) of $\arg\max_{\succsim} A \neq \emptyset \implies C(A) = \arg\max_{\succsim} A \neq \emptyset$.

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

Proof

$\Longleftarrow$: (if) Define $C : \mathbf{2}^X \to \mathbf{2}^X$ such that $C(A) = \arg\max_{\succsim} A \; \forall A \in \mathbf{2}^X$.

- WTS: C satisfies HARP.

  Take any x, y such that $\{x, y\} \subseteq A \cap B$, $x \in C(A)$, and $y \in C(B)$.

  As $y \in A$ and $x \in C(A) = \arg\max_{\succsim} A$, then $x \succsim y$; via symmetric argument, $y \succsim x$.

  From property (iii) of $\arg\max_{\succsim}$,

  $$x \sim y \text{ and } \{x, y\} \cap \arg\max_{\succsim} E = C(E) \iff \{x, y\} \subseteq \arg\max_{\succsim} E = C(E).$$

  With $E = A, B$, obtain $x \in C(B), y \in C(A)$.

□

Connecting Choice and Preferences: Revealed Preference

Theorem

Let X be a finite set. A choice function $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfies HARP if and only if there is a preference relation $\succsim \subseteq X \times X$ such that $C(A) = \arg\max_{\succsim} A \ \forall A \in \mathbf{2}^X$.

Revealed preference: obtaining $\succsim$ from C (and vice-versa)

Pins down *exactly* what choices need to satisfy to be represented by $\arg\max_{\succsim}$

Connecting Choice and Preferences: Sen's α and β

Definition

Property α. If $x \in B \subseteq A \subseteq X$ and $x \in C(A)$, then $x \in C(B)$.

α: if you choose raspberry jam when you can choose between {raspberry, strawberry, blueberry, orange}, then you choose it too when you only {raspberry, strawberry} are available. (IIA for choices)

IIA may fail: e.g., limited consideration sets, inattention, search costs and order, etc.

Definition

Property β. If $B \subseteq A \subseteq X$, $x, y \in C(B)$, and $y \in C(A)$, then $x \in C(A)$.

β: expansion consistency

Connecting Choice and Preferences: Sen's α and β

Proposition

(i) Sen's α is equivalent to the following property: if $B \subseteq A$, then $B \cap C(A) \subseteq C(B)$.

(ii) Sen's β is equivalent to the following property: if $B \subseteq A$ and $C(A) \cap C(B) \neq \emptyset$, then $C(B) \subseteq C(A)$.

(iii) HARP is equivalent to Sen's α and β.

(Exercise in lecture notes)

Overview

Utility Representation

We found a way to go from choice to preference maximisation (and back)

Now: from preference maximisation to utility maximisation (and back)

Definition

A utility function $u : X \to \mathbb{R}$ represents $\succsim \subseteq X \times X$ if $x \succsim y \iff u(x) \geq u(y), \forall x, y \in X$.

Definition

Let $\succsim \subseteq X^2$ and let $\succ$ and $\sim$ denote its asymmetric and symmetric parts.

- $A_{\succsim x} := \{y \in A \mid y \succsim x\}$ ('weakly preferred to x',);
- $A_{\succ x} := \{y \in A \mid y \succ x\}$ ('strictly preferred to x');
- $A_{x \succsim} := \{y \in A \mid x \succsim y\}$ ('weakly less preferred than x');
- $A_{x \succ} := \{y \in A \mid x \succ y\}$ ('strictly less preferred than x'); and
- $A_{x \sim} := \{y \in A \mid x \sim y\}$ ('indifferent wrt x').

Utility Representation: Finite Case

Proposition

Let X be finite. $\succsim \subseteq X^2$ is a preference relation if and only if it admits a utility representation u.

Proof

The "if" part is straightforward. For the "only if" part, define $u(x) := |X_{x\succsim}|$.

$\forall x : x \succsim y, X_{y\succsim} \subseteq X_{x\succsim}$; hence $u(x) \geq u(y)$.

If $\neg(x \succsim y)$, completeness implies $y \succ x$ and transitivity implies $X_{x\succsim} \subseteq X_{y\succsim}$.

Then, $y \succsim y \implies y \in X_{y\succsim}$ and $y \succ x \implies y \notin X_{x\succsim}$.

$\implies X_{x\succsim} \subsetneq X_{y\succsim}$ and so $u(y) > u(x)$.

□

Utility Representation: Finite Case

Proposition

Let X be finite. $\succsim \subseteq X^2$ is a preference relation if and only if it admits a utility representation u.

Note: u-representation **not** unique: for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, u represents a preference relation $\succsim$ on X iff $v := f \circ u$ does too.

But...

Proposition

(i) If $\succsim, \hat{\succsim} \subseteq X^2$ and $\succsim \neq \hat{\succsim}$, then they cannot be represented by the same utility function u.

(ii) Utility representations are unique up to positive monotone transformations.

Utility Representation: Countable Case

Can we go beyond finite set of alternatives? If X not finite, $u(x) := |X_{x\succsim}|$ doesn't work anymore... Still

Proposition

Let X be countable. $\succsim \subseteq X^2$ is a preference relation if and only if it admits a utility representation u.

Utility Representation: Countable Case

Proposition

Let X be countable. $\succsim \subseteq X^2$ is a preference relation if and only if it admits a utility representation u.

Proof

The "if" part is again straightforward. For the "only if" part, fix an order on $X = \{x_1, x_2, ...\}$ (countable X, bijection to $\mathbb{N}$). Define

$$u(x) := \sum_{n \in \left\{m \mid x_m \in X_{x\succsim}\right\}} 2^{-n}.$$

X countable $\implies$ u well-defined, sum is finite.

$\forall x : x \succsim y, X_{y\succsim} \subseteq X_{x\succsim}$; hence $u(x) \geq u(y)$.

If $\neg(x \succsim y)$, completeness implies $y \succ x$; transitivity implies $X_{x\succsim} \subseteq X_{y\succsim}$, which implies $u(y) \geq u(x)$.

Note $y = x_m$ for some $m \in \mathbb{N}$; hence, $u(y) \geq u(x) + 2^{-m} > u(x)$.

□

Utility Representation: General Case

Can we go beyond countable set of alternatives? If X not countable, $u(x) := \sum_{n \in \{m \mid x_m \in X_{x\succsim}\}} 2^{-n}$ doesn't work anymore...

Example: Lexicographic Preferences

Let $X = \mathbb{R}^2$ and define $\succsim \subseteq X$ s.t. $x \succsim y$ if $x_1 > y_1$ or ($x_1 = y_1$ and $x_2 \geq y_2$).

NB: $\succsim$ *is* complete and transitive (show it!), but... admits no utility representation!

Suppose it did, $u : X \to \mathbb{R}$.

(i) $\forall r \in \mathbb{R}$: $u(r, 1) > u(r, 0) \because (r, 1) \succ (r, 0)$.

(ii) $\forall r' > r$, $u(r', 0) > u(r, 1)$.

(iii) Hence $u(r', 1) > u(r', 0) > u(r, 1) > u(r, 0)$.

(iv) Then $\{(u(r, 0), u(r, 1)) \mid r \in \mathbb{R}\}$ is an uncountable collection of nonempty and disjoint open intervals.

(v) For any $r \in \mathbb{R}$, $(u(r, 0), u(r, 1))$ is nonempty and open.

(vi) $\mathbb{Q}$ is dense in $\mathbb{R} \implies$ for each $r \in \mathbb{R}$, $\exists$ rational number $q_r \in (u(r, 0), u(r, 1))$ s.t. $q_r \neq q_{r'}$ for $r \neq r'$.

(vii) There must be uncountably many $\{q_r\}_{r \in \mathbb{R}} \subseteq \mathbb{Q}$ but $\mathbb{Q}$ is countable: a contradiction.

Utility Representation: General Case

What goes wrong? 'Too many' indifference sets: every point in $\mathbb{R}^2$ is a different indifference set and we want to represent every indifference set with a real number.

(Note that if $\succsim$ is lexicographic by $X = \mathbb{Q}^2$, we'd be fine)

How to solve this? Avoid the problem altogether: assume that there are 'fewer' indifference sets

Definition

Let $\succsim \subseteq X^2$. A subset $X^* \subseteq X$ is **order-dense** in X with respect to $\succsim$ (or $\succsim$-dense) if, for every $x, y \in X : x \succ y$, there is $z \in X^*$ such that $x \succsim z \succ y$.

Theorem

$\succsim \subseteq X^2$ is a preference relation and $\exists$ countable $\succsim$-dense $X^* \subseteq X$ if and only if $\succsim$ admits a utility representation.

This is *exactly* the right condition: if and only if, a characterisation!

Utility Representation: General Case

Theorem

$\succsim \subseteq X^2$ is a preference relation and $\exists$ countable $\succsim$-dense $X^* \subseteq X$ iff $\succsim$ admits a utility representation.

Proof

$\Longrightarrow$: (only if) Fix an order on $X^* = \{x_1^*, x_2^*, ...\}$. Define $u(x) := \sum_{n \in \left\{m \mid x_m \in X^*_{x\succsim}\right\}} 2^{-n}$.

As X^* is countable, u is well-defined as the sum is finite.

1. WTS $x \succsim y \implies u(x) \geq u(y)$.

   $X_{y\succsim} \subseteq X_{x\succsim}$ (transitivity) $\implies X^*_{y\succsim} = (X_{y\succsim} \cap X^*_{y\succsim}) \subseteq (X_{x\succsim} \cap X^*) = X^*_{x\succsim} \implies u(x) \geq u(y)$.

2. WTS $\neg(x \succsim y) \implies u(y) > u(y)$.

   (i) $\neg(x \succsim y) \implies y \succsim x$ (completeness)

   (ii) (as before) $y \succsim x \implies X^*_{x\succsim} \subseteq X^*_{y\succsim} \implies u(y) \geq u(x)$

   (iii) (X^* $\succsim$-dense in X and $y \succ x$) $\implies \exists x_m^* : x_m^* \in X^*_{y\succsim}$ and $x_m^* \notin X^*_{x\succsim}$.

   (iv) Conclude: $u(y) \geq u(x) + 2^{-m} > u(x)$.

Utility Representation: General Case

Theorem

$\succsim \subseteq X^2$ is a preference relation and $\exists$ countable $\succsim$-dense $X^* \subseteq X$ iff $\succsim$ admits a utility representation.

Proof

$\Longleftarrow$: (if) Let $u : X \to \mathbb{R}$ be a utility representation of $\succsim$: $u(x) \geq u(y) \iff x \succsim y$.

- $\succsim$ is complete and transitive:

1. Complete: $\forall x, y \in X, (u(x) \geq u(y)$ or $u(y) \geq u(x)) \iff (x, y) \in \succsim$ or $(y, x) \in \succsim$.
2. Transitive: $x \succsim y \succsim z \iff u(x) \geq u(y) \geq u(z) \implies u(x) \geq u(z) \iff x \succsim z$.

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Proof

$\Longleftarrow$: (if) Let $u : X \to \mathbb{R}$ be a utility representation of $\succsim$: $u(x) \geq u(y) \iff x \succsim y$.

- Construct countable, $\succsim$-dense $X^* \subseteq X$.

Let $u(X) := \{u(x) \in \mathbb{R} \mid x \in X\}$.

1. For every $(p, q) \in \mathbb{Q}^2$ s.t. $p < q$ and $(p, q) \cap u(X) \neq \emptyset$, take one $x_{p,q} \in X$ s.t. $u(x_{p,q}) \in (p, q)$. Define $X_{p,q} := \{x_{p,q}\}$.
2. For every $p \in \mathbb{Q}$ s.t. $\exists x \in X : u(x) = \inf([p, \infty) \cap u(X))$, take one x_p s.t. $u(x_p) = \inf([p, \infty) \cap u(X))$., and define $X_p := \{x_p\}$.
3. By construction, $\cup_{(p,q)\in\mathbb{Q}^2 : p<q} X_{p,q}$ and $\cup_{p\in\mathbb{Q}} X_p$ are countable subsets of X $\implies X^* := \left(\cup_{p\in\mathbb{Q}} X_p\right) \cup \left(\cup_{(p,q)\in\mathbb{Q} \mid p<q} X_{p,q}\right)$ is a countable subset of X.

Utility Representation: General Case

Theorem

$\succsim \subseteq X^2$ is a preference relation and $\exists$ countable $\succsim$-dense $X^* \subseteq X$ iff $\succsim$ admits a utility representation.

Proof

$\Longleftarrow$: (if) Let $u : X \to \mathbb{R}$ be a utility representation of $\succsim$: $u(x) \geq u(y) \iff x \succsim y$.

• Construct countable, $\succsim$-dense $X^* \subseteq X$.

Let $u(X) := \{u(x) \in \mathbb{R} \mid x \in X\}$.

4. WTS X^* $\succsim$-dense in X: take any $x, y \in X : x \succ y$.

   (i) If $\exists z \in X : x \succ z \succ y \iff u(x) > u(z) > u(y)$, then

$$\begin{aligned} u(x) > u(z) > u(y) &\implies \exists p, q \in \mathbb{Q} : u(x) \geq q \geq u(z) \geq p > u(y), \quad \text{and } p < q \\ &\implies (p, q) \cap u(X) \neq \emptyset \\ &\implies \exists x_{p,q} \in X^* \subseteq X : u(x) > u(x_{p,q}) > u(y) \\ &\implies x \succsim x_{p,q} \succ y. \end{aligned}$$

Utility Representation: General Case

Theorem

$\succsim \subseteq X^2$ is a preference relation and $\exists$ countable $\succsim$-dense $X^* \subseteq X$ iff $\succsim$ admits a utility representation.

Proof

$\Longleftarrow$: (if) Let $u : X \to \mathbb{R}$ be a utility representation of $\succsim$: $u(x) \geq u(y) \iff x \succsim y$.

- Construct countable, $\succsim$-dense $X^* \subseteq X$.

Let $u(X) := \{u(x) \in \mathbb{R} \mid x \in X\}$.

4. WTS X^* $\succsim$-dense in X: take any $x, y \in X : x \succ y$.

 (ii) If $\nexists z \in X : x \succ z \succ y$.

$$\exists p \in \mathbb{Q} : u(x) > p > u(y) \text{ and } u(x) = \inf([p, \infty) \cap u(X)) \implies \exists x_p \in X^* : u(x_p) = u(x)$$
$$\implies u(x) = u(x_p) > u(y)$$
$$\implies x \succsim x_p \succ y.$$

□

Choice, Preferences, and Utility

What we've done: **choice as optimisation**

$C(A) = \arg\max_{\succsim} A = \arg\max_{x \in A} u(x)$

How restrictive is that?

Why optimisation?

Choices adapted to environment, identify mechanisms and forces at play through comparative statics, restrictions as constraints

Disciplined model of behaviour

Choice Theory and Optimisation

Let $f : X \to \mathbb{R}$ and define, for every $A \in \mathbf{2}^X$,

$$\max_{x \in A} f(x) := \{f(x) \mid x \in A \text{ and } f(x) \geq f(y), \forall y \in A\} \quad \text{and}$$

$$\arg\max_{x \in A} f(x) := \{x \in A \mid f(x) \geq f(y), \forall y \in A\}$$

Choice theory delivers useful properties for optimisation without needing to know much about the function or set over which we are optimising:

Proposition

The following properties hold:

(i) If $B \subseteq A \subseteq X$, then for any $x \in \arg\max_{z \in A} f(z)$ and $y \in \arg\max_{z \in B} f(z)$, $f(x) \geq f(y)$.

(ii) For any nonempty $A \subseteq X$ and X is finite, $\arg\max_{x \in A} f(x) \neq \emptyset$.

(iii) For $x, y \in A \subseteq X$, $f(x) = f(y)$ and $\{x, y\} \cap \arg\max_{z \in A} f(z) \neq \emptyset$ if and only if $\{x, y\} \subseteq \arg\max_{z \in A} f(z)$.

(iv) If $x \in B \subseteq A \subseteq X$, and $x \in \arg\max_{z \in A} f(z)$, then $x \in \arg\max_{z \in B} f(z)$.

(You can prove this directly with what you learned.)

Limited Observability

Example

Suppose $X = \{x, y, z\}$ and data is: $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$. HARP (and Sen's α and β) trivially satisfied, but $\nexists$ preference relation consistent with $C(A) = \arg\max_{\succsim} A$ for $A \in \{\{x, y\}, \{y, z\}, \{x, z\}\}$.

Data, in reality, is limited and we won't almost ever see 2^X.

Observing all doubletons is not enough to pin-down preference relation.

What about all triples?

With general dataset, what can we say?

Definition

Let $\mathcal{D} = \{(A, C(A)), A \in Y\}$ be a dataset with $Y \subseteq 2^X$ and C a choice function on Y.

- x **directly revealed preferred** to y if $\exists A \in Y : x \in C(A)$ and $y \in A$.
- x is **revealed preferred** to y if $\exists \{x_m\}_{m=1,\ldots,M}$ s.t. $x = x_1$, $y = x_M$ and for $i = 1, \ldots, M-1$, x_i is directly revealed preferred to x_{i+1}.
- x **revealed strictly preferred** to y if $\exists A : x \in C(A)$ and $y \in A \setminus C(A)$.

Limited Observability

Definition (GARP)

Let $\mathcal{D} = \{(A, C(A)), A \in Y\}$ be a dataset with $Y \subseteq \mathbf{2}^X$ and C a choice function on Y. $\mathcal{D}$ satisfies the **Generalised Axiom of Revealed Preference** (GARP) iff $\nexists x, y \in X$ s.t. x is revealed preferred to y and y is revealed strictly preferred to x.

Theorem

Let $\mathcal{D} = \{(A, C(A)), A \in Y\}$ be a dataset with $Y \subseteq \mathbf{2}^X$ and C a choice function on Y. $\mathcal{D}$ satisfies GARP if and only if there is a preference relation $\succsim \subseteq X^2$ such that $C(A) = \arg\max_{\succsim} A$ for any $A \in Y$.

Proof details in the notes.

Overview

1. Why Economic Theory
2. Choice and Preferences
3. Preferences and Utility
4. Limited Observability
5. More

More

- More on finite data and GARP: see notes.
- Representation of incomplete preferences: Ok (2004 JET), Eliaz & Ok (2006 GEB);
  Choice deferral: Gerasimou (2018 EJ), Pejsachowicz & Toussaert (2017 JET);
  Experiments: Halevy, Walker-Jones, & Zrill (2023 WP), Nielsen & Rigotti (2024 WP).
  (* Comments on 'incompleteness')
- Flexibility and Temptation: Kreps (1979 Ecta), Gul & Pesendorfer (2001 Ecta).
- Search: Manzini & Mariotti (2007 AER), Caplin & Dean (2011 TE), Masatlioglu Nakajima (2013 TE).
- Attention: Masatlioglu, Nakajima, & Ozbay (2012 AER).